VARIATIONS ON AN INEQUALITY FROM IMO'2001

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Abstract. Some extensions of an inequality from IMO'2001 are proven by means of the Lagrange multiplier criterion.

1. Introduction

This paper is a continuation of [1, 2], where some natural generalizations of Problem 2 from IMO'2001 have been proved. Oir aim here is to consider some other extensions of the same problem which states:

Prove that

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ac}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1,$$

where a, b and c are arbitrary positive numbers.

Many different proofs of this inequality were given during the Olympiad and it was also shown by the first author that the following more general inequality holds:

$$\frac{a}{\sqrt{a^2+\lambda bc}}+\frac{b}{\sqrt{b^2+\lambda ac}}+\frac{c}{\sqrt{c^2+\lambda ab}}\geq \frac{3}{\sqrt{1+\lambda}} \tag{1}$$

for arbitrary $a, b, c > 0$ and $\lambda \geq 8$. It is easy to see that the latter inequality is not true for $0 < \lambda < 8$. Moreover, it can be shown that in this case the infimum of the function in the left-hand side of (1) (when a, b and c run over all positive numbers) is equal to 1. This fenomenon led us to consider the following general problem:

Find the infimum and the supremum of the function

$$F_\alpha(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} \frac{1}{(1+x_i)^\alpha}$$

on the set

$$H_\lambda = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n | x_1 x_2 \dots x_n = \lambda^n, x_1, x_2, \dots, x_n > 0\},$$

where $\lambda > 0$ and α are given real constants.

2. The infimum of F_α

We shall find the infimum of the function F_α on the set H_α by means of the well-known Lagrange multiplier criterion. The next proposition has been proved in [2], but we include it here to make the paper self-contained.

Proposition 1. *For any $\alpha \in (0,1]$ we have*

$$\inf_{H_\lambda} F_\alpha = \min(1, \frac{n}{(1+\lambda)^\alpha}).$$

Proof. Suppose first that $d := \inf\limits_{H_\lambda} F_\alpha$ is not attained at a point of H_λ. Then, $d = F_\alpha = \lim_{k\to\infty} F_\alpha(x_1^{(k)}, \dots, x_n^{(k)})$, where, for example, $\lim\limits_{k\to\infty} x_n^{(k)} = 0$ or $+\infty$. Hence, for example, $\lim\limits_{k\to\infty} x_1^{(k)} = +\infty$ or 0 and in both cases we see that $d \geq 1$. Note that if $\lim\limits_{k\to\infty} x_s^{(k)} = +\infty$ for $s = 1, 2 \dots, n-1$ and $\lim\limits_{k\to\infty} x_n^{(k)} = 0$, then $\lim\limits_{k\to\infty} F_\alpha(x_1^{(k)}, \dots, x_n^{(k)}) = 1$. Now, let d is attained at a point of H_λ. Consider the function

$$F(x_1, x_2, \dots, x_n) = F_\alpha(x_1, x_2, \dots, x_n) + \mu(x_1 x_2 \dots x_n - \lambda^n).$$

Then the Lagrange multiplier criterion says that d is attained at a point $(x_1, x_2, \dots, x_n) \in H_\lambda$ such that

$$\frac{\partial F}{\partial x_i} = -\frac{\alpha}{(1+x_i)^{\alpha+1}} + \frac{\mu x_1 \dots x_n}{x_i} = 0,$$

i.e., when

$$\frac{x_i}{(1+x_i)^{\alpha+1}} = \frac{x_j}{(1+x_j)^{\alpha+1}}, 1 \leq i, j \leq n. \tag{2}$$

Consider the function $g(x) = \dfrac{x}{(1+x)^{\alpha+1}}$. Then, $g'(x) = \dfrac{1-\alpha x}{(1+x)^{\alpha+2}}$, and, therefore, $g(x)$ takes each its value at most twice. Hence (2) shows that $x_1 = \dots = x_k = x$ and $x_{k+1} = \dots = x_n = y$ for some $1 \leq k \leq n$. If $k = n$, then $x_1 = x_2 = \dots = x_n = \lambda$ and $F_\alpha(x_1, x_2, \dots, x_n) = \dfrac{n}{(1+\alpha)^\lambda}$. If $k < n$, then

$$F_\alpha(x_1, x_2, \dots, x_n) = \frac{k}{(1+x)^\alpha} + \frac{n-k}{(1+y)^\alpha} \geq \frac{1}{(1+x)^\alpha} + \frac{1}{(1+y)^\alpha}.$$

To prove Proposition 1 it is sufficient to show that

$$\frac{1}{(1+x)^\alpha} + \frac{1}{(1+y)^\alpha} > 1 \tag{3}$$

provided

$$\frac{x}{(1+x)^{\alpha+1}} = \frac{y}{(1+y)^{\alpha+1}}, x \neq y. \tag{4}$$

Set $\beta = \dfrac{1}{\alpha} \geq 1, z = (1+x)^\alpha$ and $t = (1+y)^\alpha$. Then (3) and (4) can be written respectively as $z + t > zt$ and $(zt)^\beta = \dfrac{z^{\beta+1} - t^{\beta+1}}{z-t}$. So, we have to prove that

$$(z+t)^\beta \geq \frac{z^{\beta+1} - t^{\beta+1}}{z-t}. \tag{5}$$

Assume that $z < t$ and set $u = \dfrac{z}{t} < 1$. Applying Bernoulli's inequality twice we obtain $(1+u)^\beta \geq 1 + \beta u > \dfrac{1-u^{\beta+1}}{1-u}$ which is just the inequality (5). $\square$

The next example shows that for a given $\alpha > 1$ a result similar to Proposition 1 could be expected only for sufficiently large n.

Example 1. *Let $\alpha = 2$ and $n = 2$. Then the function $F_2(x_1, x_2)$ attains minimum on H_λ given by*

$$\min_{H_\lambda} F_2 = \begin{cases} \dfrac{2}{(1+\lambda)^2} & \text{if } \lambda \geq \frac{1}{2} \\ \dfrac{1-2\lambda^2}{(1-\lambda^2)^2} & \text{if } 0 < \lambda \leq \frac{1}{2}. \end{cases} \tag{6}$$

Proof. To prove (6) we proceed as in the proof of Proposition 1. First note that if $x_1 \to 0$ or $+\infty$, then $x_2 \to +\infty$ or 0 and, in both cases, $F_2(x_1, x_2) \to 1$. Consider the points $(x_1, x_2) \in H_\lambda$ such that

$$\frac{x_1}{(1+x_1)^3} = \frac{x_2}{(1+x_2)^3}. \tag{7}$$

If $x_1 = x_2 = \lambda$, then $F_2(x_1, x_2) = \dfrac{2}{(1+\lambda)^2}$. If $x_1 \neq x_2$, then (7) is equivalent to $x_1 + x_2 = \dfrac{1}{\lambda^2} - 3$. This together with $x_1 x_2 = \lambda^2$ implies that $\dfrac{1}{\lambda^2} - 3 \geq 2\lambda$, i.e., $\lambda < \dfrac{1}{2}$ and $F_2(x_1, x_2) = \dfrac{1-2\lambda^2}{(1-\lambda^2)^2}$. Hence (6)

follows from the inequalities $\frac{1-2\lambda^2}{(1-\lambda^2)^2} < 1$ and $\frac{1-2\lambda^2}{(1-\lambda^2)^2} < \frac{2}{(1+\lambda)^2}$ for any $\lambda > 0$, and $\frac{2}{(1+\lambda)^2} < 1$ for $\lambda \geq \frac{1}{2}$. □

The next proposition gives a partial result in the case $\alpha > 1$.

Proposition 2. *For any $\alpha > 1$ and any integer $n \geq \alpha + 1$ we have*

$$\inf_{H_\lambda} F_\alpha = \min(1, \frac{n}{(1+\lambda)^\alpha}).$$

Proof. Proceeding as in the proof of Proposition 1 it is sufficient to prove that

$$(1+(n-1)u)^\beta > \frac{1-u^{\beta+1}}{1-u}$$

for $\beta = \frac{1}{\alpha} < 1$ and $0 < u < 1$. Since $n-1 \geq \alpha$ we have $1+(n-1)u \geq 1 + \frac{n}{\beta}$ and it is enough to show that

$$(1+\frac{u}{\beta})^\beta > \frac{1-u^{\beta+1}}{1-u} \tag{8}$$

for $\beta, u \in (0,1)$. Consider the function

$$f(x) = (1-x)(1+\frac{x}{\beta})^\beta + x^{\beta+1} - 1 \text{ for } x \in [0,1].$$

Since

$$f'(x) = \frac{(1+\beta)x}{\beta}(\beta x^{\beta-1} - (1+\frac{x}{\beta})^{\beta-1})$$

the equation $f'(x) = 0$ has a unique real root $x_0 = (\beta^{\frac{1}{\beta-1}} - \beta^{-1})^{-1}$. On the other hand, since $f(0) = f(1) = 0$ and $\beta - 1 < 0$, it follows that $x_0 \in (0,1)$, $f'(x) > 0$ for $x \in (0, x_0)$ and $f'(x) < 0$ for $x \in (x_0, 1)$. Hence $f(x) > 0$ for $x \in (0,1)$ and the inequality (8) is proved. □

Remark 1. As Example 1 suggests, if $\alpha > 1$ and $n < \alpha + 1$, then a result similar to Proposition 2 is not true. The authors do not know the value of $\inf_{H_\lambda} F_\alpha$ for such α and n.

To complete this section it remains to consider the case $\alpha < 0$.

Proposition 3. *For any $\alpha < 0$ the function $F_\alpha(x_1, \dots, x_n)$ attains minimum on H_λ given by*

$$\min_{H_\lambda} F_\alpha = \frac{n}{(1+\lambda)^\alpha}.$$

Proof. We may proceed as in the proof of Proposition 1 but in this case the statement follows directly from the fact that the function $f(x) = \dfrac{1}{(1+e^x)^\alpha}$ is convex for $\alpha < 0$ since $f''(x) > 0$. □

3. The supremum of F_α

The results obtained in this section are dual analogs of that in Section 2.

Proposition 4. *For any $\alpha \geq 1$ we have*

$$\sup_{H_\lambda} F_\alpha = \max(n-1, \frac{n}{(1+\lambda)^\alpha}).$$

Proof. We proceed as in the proof of Proposition 1. If $\sup\limits_{H_\lambda} F_\alpha$ is not attained at a point of H_λ then we may assume that $x_n \to +\infty$ and obviously we have $\sup\limits_{H_\lambda} F_\alpha \leq n-1$. Note also that if $x_1 \to 0, \dots, x_{n-1} \to 0$ and $x_n \to +\infty$, then $F_\alpha(x_1, \dots, x_n) \to n-1$.

Next consider the case when $\sup\limits_{H_\lambda} F_\alpha$ is attained at a point of H_λ such that $x_1 = \dots = x_k = x$ and $x_{k+1} = \dots = x_n = y$. If $x = y$, then $x_1 = \dots = x_n = \lambda$ and $F_\alpha(x_1, \dots, x_n) = \dfrac{n}{(1+\lambda)^\alpha}$. If $x \neq y$, then $k < n$ and $F_\alpha(x_1, \dots, x_n) = \dfrac{k}{(1+x)^\alpha} + \dfrac{n-k}{(1+y)^\alpha}$. So, it is enough to prove that if $\dfrac{x}{(1+x)^{\alpha+1}} = \dfrac{y}{(1+y)^{\alpha+1}}$ and $x < y$, then $\dfrac{n-1}{(1+x)^\alpha} + \dfrac{1}{(1+y)^\alpha} < n-1$. But this follows from the inequality $\dfrac{1}{(1+x)^\alpha} + \dfrac{1}{(1+y)^\alpha} < 1$, which can be proved by using Bernoulli's inequality for $\beta = \dfrac{1}{\alpha} < 1$ as in the proof of Proposition 1. □

The next example is dual to Example 1.

Example 2. *Let* $\alpha = \dfrac{1}{2}$ *and* $n = 2$*. Then the function* $F_{\frac{1}{2}}(x_1, x_2)$ *attains maximum on* H_λ *given by*

$$\max_{H_\lambda} F_{\frac{1}{2}} = \begin{cases} \dfrac{\lambda}{\sqrt{\lambda^2 - 1}} & \textit{if } \lambda > 2 \\ \dfrac{2}{\sqrt{1+\lambda}} & \textit{if } 0 < \lambda \le 2. \end{cases} \tag{9}$$

Proof. First note that if $x_1 \to 0$ or $+\infty$, then $x_2 \to +\infty$ or 0, and, in both cases, $F_{\frac{1}{2}}(x_1, x_2) \to 1$. Now consider the points $(x_1, x_2) \in H_\lambda$ for which $\dfrac{x_1}{(1+x_1)^{\frac{3}{2}}} = \dfrac{x_2}{(1+x_2)^{\frac{3}{2}}}$. If $x_1 = x_2$, then $F_{\frac{1}{2}}(x_1, x_2) = \dfrac{2}{\sqrt{1+\lambda}}$. If $x_1 \neq x_2$, then $x_1 + x_2 = \lambda^2(\lambda^2 - 3)$ and since $x_1 x_2 = \lambda^2$ we have $\lambda > 2$ and $F_{\frac{1}{2}}(x_1, x_2) = \dfrac{\lambda}{\sqrt{\lambda^2 - 1}}$. Hence (9) follows from the inequalities $\dfrac{\lambda}{\sqrt{\lambda^2 - 1}} > 1$ and $\dfrac{\lambda}{\sqrt{\lambda^2 - 1}} \ge \dfrac{2}{\sqrt{1+\lambda}}$ for $\lambda > 2$, and $\dfrac{2}{\sqrt{1+\lambda}} > 1$ for $\lambda \le 2$. □

The dual analog of Proposition 2 is the following

Proposition 5. *For any* $\alpha \in (0, 1)$ *and any integer* $n \ge \dfrac{1}{\alpha} + 1$ *we have*

$$\sup_{H_\lambda} F_\alpha = \max(n - 1, \frac{n}{(1+\lambda)^\alpha}).$$

Proof. Proceedings as in the proof of Proposition 2 it is enough to show that $(1 + \dfrac{u}{\beta})^\beta < \dfrac{1 - u^{\beta+1}}{1 - u}$ for arbitrary $u \in (0, 1)$ and $\beta > 1$. This can be done in the same way as the proof of the inequality (8). □

Finally, note that in the case $\alpha < 0$ obviously the supremum of F_α is equal to $+\infty$.

Remark 2. As Example 2 suggests, if $\alpha \in (0, 1)$ and $n < \dfrac{1}{\alpha} + 1$, then a result similar to Proposition 5 is not true. The authors do not know the value of $\sup\limits_{H_\lambda} F_\alpha$ for such α and n.

References

[1] O. Mushkarov & N. Nikolov, *A generalization of an inequality from IMO'2001* (in Bulgarian), Math. Plus 3 (2002), 17-18.

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